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Exact solitary wave solutions of non-linear evolution and wave equations using a direct algebraic method

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Abstract. We present a systematic and formal approach toward finding solitary wave solutions of non-linear evolution and wave equations from the real exponential solutions of the underlying linear equations. The physical concept is one of the mixing of these elementary solutions through the non-linearities in the system. In the present paper the emphasis is, however, on the mathematical aspects, i.e. the formal procedure necessary to find single solitary wave solutions. By means of examples we show how various cases of pulse-type and kink-type solutions are to be obtained by this method. An exhaustive list of equations so treated is presented in tabular form, together with the particular intermediate relations necessary for deriving their solutions. We also outline the extension of our technique to construct N-soliton solutions and indicate connections with other existing methods.

1. Introduction

In an earlier paper (Hereman *et al* 1985) we presented a physical approach to constructing solitary wave solutions of non-linear evolution and wave equations from the mixing of the real exponential travelling wave solutions of the underlying linear equation. This was motivated by the knowledge of the nature of the final closed form solution, which can be expanded into an infinite series of the harmonics of the real exponential solution(s) of the linear dispersive system. The feasibility of this approach lies in the fact that, in a mathematical sense, it is far simpler to 'mix' real exponentials rather than harmonic functions built up from imaginary exponentials, which, incidentally, are also solutions of the linear system.

The present paper develops a more mathematically rigorous and systematic procedure for deriving solitary wave solutions of various non-linear partial differential

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equations (PDEs), while still not losing track of the physical concepts (evolution by mixing of linear solutions) which were the essence of our earlier paper. In other words we want to offer a physically transparent straightforward step-by-step technique—along with necessary justifications for every step—for constructing single solitary wave solutions relevant to applications in various areas of physics and engineering. In this regard, the present paper supplements the heuristic approach for deriving such non-linear dispersive equations (Korpel and Banerjee 1984) in the sense that it enables the researcher to examine the final solution for applicability as well as to assess whether or not his particular system is likely to exhibit travelling solitary wave behaviour.

We do not claim that the formal mathematical aspects of our treatment have never been proposed before. However, we do believe that our physical interpretation (although, of course, implicit in the mathematics) is novel. What is perhaps more important, we find that it greatly aids in understanding the problem in terms of the physical phenomena the engineer/scientist is familiar with.

As regards formal mathematical approaches, Sawada and Kotera (1974), Rosales (1978), Whitham (1979), Wadati and Sawada (1980a, b) and Hickernell (1983) have all employed a perturbation technique. Their methods lead, after some tedious algebraic manipulations, to the known N-solitary wave solutions of, for instance, the KdV, mKdV, Burgers and Boussinesq equations. Their iterative procedures are closely linked to Hirota's direct method (Hirota 1980) for finding multisoliton solutions, and to the summation procedure of the Padé type described by Turchetti (1980) and Liverani and Turchetti (1983). Rigorous mathematical analyses of the problem of direct linearisation of non-linear PDEs have recently been presented by Taflin (1983) and Santini *et al* (1984). Pöppe (1983, 1984) has used the Fredholm determinant method in a new and rigorous way to investigate soliton equations and to construct their solutions. Pöppe, Rosales, and Wadati and Sawada point out the connection between their techniques and, for instance, the Bäcklund transformation technique (Miura 1976) and the inverse scattering method (Ablowitz and Segur 1981) focusing on a novel derivation of the Gel'fand-Levitan-Marčenko equations. The mathematical formulation (though not the interpretation) of Korpel's real exponential approach (Korpel 1978), from which we derive our method, is closely related to the Rosales' perturbation technique and the trace method of Wadati and Sawada (1980a, b). The reader, however, should bear in mind that the word 'perturbation' is a misnomer since, e.g., Rosales' technique not only leads to exact solutions, but it essentially implies the principle of harmonic generation and mixing starting out from real exponential solutions of the linear equation, as explicitly assumed in Korpel's approach.

In § 2 we discuss the general solution method itself to develop closed-form single solitary wave solutions of the pulse or kink types which may have a constant in their expansion into real exponentials. Throughout this section we formalise the procedure and give elaborate guidelines for solving the recursion relations that occur.

In § 3 we exemplify our technique by first applying it to the mKdV equation. We then treat the KdV equation with an additional fifth-order dispersion term (already mentioned by Zabusky (1967)). This example shows how to proceed in the case of multiple solutions of the linear equation. Thirdly, we deal with the non-linear Klein-Gordon equation and show how the extension of the technique leads to both the pulse-type and the kink-type solutions.

We furthermore find a kink-type solution of the sine-Gordon equation (Pöppe 1983) by first decomposing it into a coupled non-linear system and employing the real exponential approach. This is done with the intention of exemplifying the applicability of our technique to coupled systems in general. Alternatively, the sine-Gordon equation may also be solved by transforming it to the non-linear Klein–Gordon equation (Gibbon *et al* 1978).

In § 4 we briefly mention other equations solved by our fairly simple method, providing appropriate references for comparison purposes, and tabulating the results obtained so far.

In § 5, we extend our concept of evolution by mixing of linear solutions to construct N-soliton solutions using the KdV equation as an illustrative example. We furthermore indicate the connections between our approach and the perturbation technique of Rosales (1978) and Whitham (1979) and the trace method of Wadati and Sawada (1980a, b).

Finally, we discuss some outstanding questions and outline the work in progress and areas of future investigations.

2. The solution method

In this section we present the details of the general solution method for exactly solving non-linear evolution and wave equations (as outlined by Hereman *et al* (1985)) leading to mainly hyperbolic-type solitary waves.

The method, which is straightforwardly applicable, goes in the following ten steps.

(a) Starting from the non-linear equation in $1+1$ dimensions with x and t as the space and time coordinates respectively, we introduce a travelling frame of reference by $\xi = x - vt$. This transforms the given non-linear PDE in $u(x, t)$ into an ODE in $\phi(\xi) \triangleq u(x, t)$. Note that v is the constant anticipated velocity of the travelling wave solution dependent on the wave amplitude, as we shall see later.

(b) We next integrate the ODE with respect to ξ, as many times as possible, but avoiding integral equations. For example, for evolution equations of the type

$$u_t = f(u, u_x, u_{2x}, \ldots, u_{nx}), \qquad n \in \mathbb{N}, \tag{1}$$

where $u_t = \partial u/\partial t$, $u_{nx} = \partial^n u/\partial x^n$, the maximum permissible number of integrations is 1. Similarly, for wave equations containing a second derivative in time, the number of integrations may not exceed 2. For reasons that will be explained in step (d), we only leave in the integration constant of the last integration.

(c) To obtain the most general solitary wave solution, possibly having a constant (dc) term c_1 in its expansion into real exponentials, we substitute

$$\phi = c_1 + \hat{\phi} \tag{2}$$

into the non-linear equation for ϕ.

(d) We now consider the linear part of the resulting equation in $\hat{\phi}$, by setting the coefficient(s) of the non-linear term(s) equal to zero or simply neglecting them. In our search for solitary wave solutions, we are motivated to look for solutions of the linear equation of the form $\exp[-K(v)\xi]$, where $K(v)$ is a real function of v. Substitution of this exponential function in the linear equation yields, in general, an equation of the form $f(K) = 0$, where $f(K)$ is a polynomial in K. The solution of this equation leads to a solution set $\{K_p(v) | p = 1, \ldots, M\}$ and requires the constant term in the linear equation to be equal to zero. This, of course, imposes certain conditions on c_1 as well as on the integration constant.

For non-dissipative evolution and wave equations, i.e. for which the dispersion relation $\omega = W(k)$, expressing the angular frequency ω as a function of the wavenumber k, is a real odd function of k (Korpel and Banerjee 1984), the polynomial for K will only have terms with even powers, so that if $K_i(v) > 0$, $i = 1, \ldots, N$, is a solution, then $-K_i(v)$ is also a solution, with $M = 2N$. In other words, for every decaying exponential in the set, there is a rising exponential. On the other hand, for equations containing dissipative terms, e.g. the Burgers equation (Whitham 1974), this is no longer true. Only one real solution for K will then exist for the linear part of the equation.

We can now define $\{g_p(\xi) \triangleq \exp(-K_p\xi): p = 1, \ldots, M\}$ to be the set of 'fundamental' functions, which we will use to arrive at the closed form solution of the given non-linear equation.

For non-dissipative equations M is an even number, say $2N$, and one has only to use one type of exponential (i.e. the decaying ones or the rising ones) (Hereman *et al* 1985).

In all the examples for non-dissipative equations given in § 3, we work with the N decaying exponentials, labelled as $g_1, \ldots, g_N$, and corresponding to the first N functions of K_p defined to be positive. For dissipative equations there is no such choice; we have to use all M linear solutions.

(e) For the sake of mathematical convenience, we may normalise a few coefficients of the non-linear terms by a simple scaling transformation of $\hat{\phi}$ into $\tilde{\phi}$.

(f) We now have to solve the non-linear equation in $\tilde{\phi}$. As explained in a previous paper (Hereman *et al* 1985), the solution to non-dissipative equations could be explicitly expressed in the form

$$\tilde{\phi} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} a_{n_1 \ldots n_N} g_1^{n_1}(\xi) \ldots g_N^{n_N}(\xi) \tag{3}$$

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} a_{n_1 \ldots n_N} \exp[-(n_1K_1 + n_2K_2 + \ldots + n_NK_N)\xi], \tag{4}$$

$$(n_1, n_2, \ldots, n_N) \neq (0, 0, \ldots, 0).$$

However, in most non-dissipative cases having $N > 1$, it appears possible to determine the highest common factor K of the set $\{K_p: p = 1, \ldots, N\}$ defined by

$$K \triangleq K_p/M_p, \qquad p = 1, \ldots, N, \tag{5}$$

where M_p is a positive integer, enabling $\tilde{\phi}$ to be re-expressed as

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n(\xi) \tag{6}$$

with

$$g(\xi) \triangleq \exp(-K\xi). \tag{7}$$

This will become clear through the example in § 3.2. Note that for dissipative equations (3)-(5) may still be used, provided N is replaced by M.

(g) We next substitute (6) with (7) into the reduced non-linear equation which, in general, will contain non-linear and dispersion and/or dissipation terms of arbitrary order and degree. Linear terms in the reduced equation may be expressed in the form $\Sigma_{n=1}^{\infty} P(n) a_n g^n$, where the degree λ of the polynomial $P(n)$ equals the highest order of dispersion (or dissipation) minus the number of integrations carried out in step (b).

The structure of $P(n)$ may be readily derived by examining terms of the form $\tilde{\phi}_{q\xi}(q \in \mathbb{N})$ which give rise to terms of the form n^q in the polynomial $P(n)$. This will, once again, become clear through the examples in § 3.

To deal with the non-linear terms, we need to employ the extension of Cauchy's (product) rule (Gradshteyn and Ryzhik 1980) for multiple series.

Lemma 1 (extension of Cauchy's product rule). If

$$F^{(i)} \triangleq \sum_{n_i=1}^{\infty} f_{n_i}^{(i)} \qquad (i=1, \ldots, I) \tag{8}$$

represents I infinite convergent series then

$$\prod_{i=1}^{I} F^{(i)} = \sum_{n=1}^{\infty} \sum_{r=I-1}^{n-1} \cdots \sum_{m=2}^{k-1} \sum_{l=1}^{m-1} f_l^{(1)} f_{m-l}^{(2)} \cdots f_{n-r}^{(I)}. \tag{9}$$

Proof. Applying Cauchy's rule to the product of $F^{(1)}$ and $F^{(2)}$ we get

$$\begin{aligned} F^{(1)}F^{(2)} &= \left(\sum_{n_1=1}^{\infty} f_{n_1}^{(1)}\right)\left(\sum_{n_2=1}^{\infty} f_{n_2}^{(2)}\right) \\ &= (f_1^{(1)}+f_2^{(1)}+\ldots)(f_1^{(2)}+f_2^{(2)}+\ldots) \\ &= (f_1^{(1)}f_1^{(2)})+(f_1^{(1)}f_2^{(2)}+f_2^{(1)}f_1^{(2)})+\ldots \\ &= \sum_{m=2}^{\infty} \sum_{l=1}^{m-1} f_l^{(1)} f_{m-l}^{(2)} \qquad (10) \\ &\triangleq \sum_{m=2}^{\infty} h_m^{(2)} \qquad (11) \end{aligned}$$

where, in (11), we have re-expressed the double product in a single series in order to re-apply Cauchy's rule. This yields

$$\begin{aligned} F^{(1)}F^{(2)}F^{(3)} &= \left(\sum_{m=2}^{\infty} h_m^{(2)}\right)\left(\sum_{n_3=1}^{\infty} f_{n_3}^{(3)}\right) \\ &= \sum_{p=3}^{\infty} \sum_{m=2}^{p-1} h_m^{(2)} f_{p-m}^{(3)} \qquad \text{(applying (10))} \\ &= \sum_{p=3}^{\infty} \sum_{m=2}^{p-1} \sum_{l=1}^{m-1} f_l^{(1)} f_{m-l}^{(2)} f_{p-m}^{(3)} \qquad \text{(using (11))} \qquad (12) \\ &\triangleq \sum_{p=3}^{\infty} h_p^{(3)}. \qquad (13) \end{aligned}$$

Proceeding step by step as above, we finally obtain (9).

For our purposes, we have to apply (9) to multiple products of $\tilde{\phi}$ and its derivatives with respect to ξ. Using (6) and (7), it is clear that for

$$F^{(i)} = \tilde{\phi}_{q\xi} = (-K)^q \sum_{n_i=1}^{\infty} n_i^q a_{n_i} g^{n_i}(\xi), \qquad q \in \mathbb{N}, \tag{14}$$

a typical non-linear term, for instance $\tilde{\phi}\tilde{\phi}_\xi^2\tilde{\phi}_{2\xi}$, may be expanded as

$$\tilde{\phi}\tilde{\phi}_\xi^2\tilde{\phi}_{2\xi}=\tilde{\phi}\tilde{\phi}_\xi\tilde{\phi}_\xi\tilde{\phi}_{2\xi}$$

$$=(-K)^4\sum_{n=4}^{\infty}\sum_{k=3}^{n-1}\sum_{m=2}^{k-1}\sum_{l=1}^{m-1} l^0(m-l)^1(k-m)^1(n-k)^2 a_l a_{m-l} a_{k-m} a_{n-k} g^n. \quad (15)$$

Hence, in general, the non-linear equation may be replaced by

$$\sum_{n=1}^{\infty} P(n) a_n g^n + c_H \sum_{n=H}^{\infty}\sum_{r=H-1}^{n-1}\cdots\sum_{m=2}^{k-1}\sum_{l=1}^{m-1}(l)^{\mu_1}(m-l)^{\mu_2}\ldots(n-r)^{\mu_H} a_l a_{m-l}\ldots a_{n-r} g^n$$

$$+\ldots+c_L\sum_{n=L}^{\infty}\sum_{r=L-1}^{n-1}\cdots\sum_{m=2}^{k-1}\sum_{l=1}^{m-1}(l)^{\nu_1}(m-l)^{\nu_2}\ldots(n-r)^{\nu_L} a_l a_{m-l}\ldots a_{n-r} g^n$$

$$=0, \quad (16)$$

where H and L refer to the highest and lowest orders of the non-linearity; μ_κ, ν_κ, etc, indicate the order of the derivative associated with each $\tilde{\phi}$ in a non-linear term, and where the powers of $(-K)$ are possibly absorbed in the constants $c_H, \ldots, c_L$, if they are not eliminated by the scaling in step (e). Since the summation in the first term starts from 1 and the summations in the non-linear terms start from L ($L \geqslant 2$), we may take one or more of the coefficients $a_1, \ldots, a_j, \ldots, a_{L-1}$ arbitrary if $P(j)=0$. On the other hand, if $P(j) \neq 0$ ($j=1, \ldots, L-1$), then one or more of the above coefficients will be zero. For $n \geqslant H$, the summation over n in (16) may be deleted to obtain the generalised recursion relation

$$P(n)a_n + c_H\sum_{r=H-1}^{n-1}\cdots\sum_{m=2}^{k-1}\sum_{l=1}^{m-1}(l)^{\mu_1}(m-l)^{\mu_2}\ldots(n-r)^{\mu_H} a_l a_{m-l}\ldots a_{n-r}$$

$$+\ldots+c_L\sum_{r=L-1}^{n-1}\cdots\sum_{m=2}^{k-1}\sum_{l=1}^{m-1}(l)^{\nu_1}(m-l)^{\nu_2}\ldots(n-r)^{\nu_L} a_l a_{m-l}\ldots a_{n-r}=0. \quad (17)$$

(h) In this crucial step we show how to solve the recursion relation (17) by simple algebraic computation. In order to construct the general form of the coefficients a_n, two straightforward techniques are available.

(1) We can compute the first few coefficients step by step, which allows us often to recognise the general structure for a_n. For well known non-linear equations like the KdV, mKdV, Burgers, the Tasso-Sharma-Olver (Olver 1977) and the non-linear Klein-Gordon (Whitham 1974) equations, this simple method is readily applicable (see Korpel 1978, Hereman *et al* 1985, and the examples in §§ 3.1, 3.3 and 3.4).

(2) For more complicated recursion relations, the general form of a_n is no longer recognisable as such.

Note, however, that a_n will contain a polynomial in n of degree

$$\delta=(\lambda-H+1-\Lambda)/(H-1), \quad (18)$$

with $\Lambda=\Sigma_{i=1}^{H}\mu_i$ and where λ and H are the degrees of the polynomial $P(n)$ and the highest order of the non-linearity respectively. Now, each of the $(H-1)$ summations appearing in (17) will raise the degree $H\delta$ of the product $a_l \ldots a_{n-r}$ by 1. Equating, therefore, the degree $(\lambda+\delta)$ of $P(n)a_n$ with the degree $(H\delta+H-1+\Lambda)$ of the highest non-linear term, equation (18) readily follows. Furthermore, to derive additional information about the structure of a_n, note that if $a_j = 0$ $(j = 1, \ldots, L-1)$, then $(n-j)$ is a factor of a_n. Also, we may observe that if $P(n) = P(-n)$, then $(n+j)$ is also a factor (see, for instance, the example in § 3.2). In addition, one can check that if the recursion relation (17) has only one non-linear term with a positive coefficient c_H, then $a_n \propto (-1)^n$, expressing alternation in sign.

While at this point, it is intriguing to speculate on how we could derive an evolution or wave equation from a desired solution by examining the structure of a_n. For instance, for a sech^2-type solution, it is required that $a_n \propto n$ (see table 1); hence, $\delta = 1$. From (18), we thus obtain

$$\lambda = (H-1)(1+\delta) + \Lambda = 2(H-1) + \Lambda. \tag{19}$$

Realising that λ is equal to the order of dispersion minus 1 for non-dissipative evolution equations, we have to combine, for the case $\Lambda = 0$, at least a quadratic, cubic and quartic non-linearity with a cubic, fifth-order and seventh-order dispersion, respectively, in the non-linear evolution equation to obtain a sech^2-hump-like solution. This can be verified by examining the KdV and its fifth- and seventh-order generalisations listed in table 1 (compare, for example, with Lax (1968)).

Finally, to obtain any remaining constant coefficients in a_n, we substitute the polynomial form for it in (17), apply the standard formulae for the sums of powers of positive integers (Spiegel 1968), and perform necessary straightforward computations, as illustrated in the example in § 3.2.

(i) In order to construct the solution $\tilde{\phi}$, we next substitute the coefficients a_n into (6). The obtained power series of decaying exponentials $g(\xi)$, given by (7), is convergent in the region $\xi > \xi_0$, where ξ_0 is a well defined positive number.

However, due to the particular form of the power series, $\tilde{\phi}$ can be written in closed form, which may be re-expanded in a convergent power series of a rising exponential $g^{-1}(\xi)$ (which is a solution of the linear equation) in the region $\xi < \xi_0$ in the case of non-dissipative equations. Since the closed form for $\tilde{\phi}$ is continuous at $\xi = \xi_0$, this represents a valid solution over the entire region $-\infty < \xi < \infty$.

For dissipative equations however, the re-expansion, though mathematically possible, has no physical interpretation, since $g^{-1}(\xi)$ is no longer a solution of the linear equation. However, as shown for the case of the Burgers equation (Hereman *et al* 1985), the closed form of $\tilde{\phi}$ yields the steady state, which is now achieved at $t \to \infty$, through a balance between a continuous supply of energy and the dissipation in the system, which is ultimately responsible for the boundedness of the closed form solution over the entire region $-\infty < \xi < \infty$.

The closed form for $\tilde{\phi}$ in each case is parametrised by a constant, which leads to an arbitrary phase shift δ.

(j) Finally, returning to the original dependent variable u and the independent variables x and t, an exact solitary wave solution of the non-linear PDE is obtained.

3. Examples

3.1. The modified Korteweg–de Vries equation

As a first example to demonstrate the method outlined in § 2, we have chosen the well known modified Korteweg–de Vries (mKdV) equation (Lamb 1980, Dodd *et al* 1982)

$$u_t + \alpha u^2 u_x + u_{3x} = 0, \tag{20}$$

where α is an arbitrary real number determined by the cubic non-linearity in a physical system. The subscripts in (20) refer to the partial derivatives of $u(x, t)$ with respect to time t and space variable x.

Introduction of the new independent variable

$$\xi = x - vt, \tag{21}$$

where v is the constant anticipated velocity of the solitary wave, transforms the PDE (20) into an ODE for $\phi(\xi) \triangleq u(x, t)$:

$$-v\phi_\xi + \alpha\phi^2\phi_\xi + \phi_{3\xi} = 0. \tag{22}$$

We next integrate (22) once with respect to ξ, to obtain

$$-v\phi + \tfrac{1}{3}\alpha\phi^3 + \phi_{2\xi} = 0, \tag{23}$$

where, for simplicity, we have not introduced the constant c_1 mentioned in step (c) in § 2.

The linear part of (23) has two real exponential solutions, denoted by $\exp[\pm K(v)\xi]$, where

$$K = v^{1/2}, \qquad v > 0, \tag{24}$$

indicating already that only solitary waves travelling to the right are possible. Note that for this example $M = 2$, $N = 1$ and $K_1 = -K_2 = K$; hence,

$$g_1(\xi) = 1/g_2(\xi) = \exp(-K\xi) \triangleq g(\xi). \tag{25}$$

We now normalise the non-linear equation (23) by a simple scaling transformation

$$\phi = (3v/\alpha)^{1/2}\tilde{\phi} \tag{26}$$

to get

$$-v\tilde{\phi} + v\tilde{\phi}^3 + \tilde{\phi}_{2\xi} = 0. \tag{27}$$

To obtain a particular solution of (27), we substitute the series expansion

$$\tilde{\phi}(\xi) = \sum_{n=1}^{\infty} a_n g^n(\xi) \tag{28}$$

into it and apply Cauchy's rule for the triple product appearing in the cubic non-linearity. This yields the recursion relation

$$(n^2 - 1)a_n + \sum_{m=2}^{n-1} \sum_{l=1}^{m-1} a_l a_{m-l} a_{n-m} = 0, \qquad n \geq 3, \tag{29}$$

where a_1 is arbitrary, but to be taken positive as we will see later, and with $a_2 = 0$. It may be readily verified that all a_n with an even label will be zero. To compare with the theory in § 2, for the present example $H = L = 3$, $\mu_\kappa = \nu_\kappa = 0$ and $c_H = c_L = \tfrac{1}{2}$.

The recursion relation (29) is readily solved. By a step-by-step calculation of the first few coefficients, resulting in

$$a_3 = -a_1^3/2^3, \tag{30}$$

$$a_5 = +a_1^5/2^6, \tag{31}$$

$$a_7 = -a_1^7/2^9, \tag{32}$$

etc, it is easy to write down a non-trivial solution satisfying (29):

$$a_{2n'} = 0, \tag{33}$$

$$a_{2n'+1} = (-1)^{n'} a_1^{2n'+1}/2^{3n'}, \qquad n' = 1, 2, 3, \ldots. \tag{34}$$

Note that a_n is a polynomial in n of degree $\delta = 0$, as confirmed by (18) since $\lambda = 2$ and $H = 3$.

Substituting (33) and (34) into (28) yields

$$\tilde{\phi} = \sum_{n'=0}^{\infty} \frac{(-1)^{n'} a_1^{2n'+1}}{2^{3n'}} (g(\xi))^{2n'+1} \tag{35}$$

$$= 2\sqrt{2} a g(\xi)/(1 + a^2 g^2(\xi)), \qquad a = a_1/2\sqrt{2}, \tag{36}$$

where we have made use of the identity (see e.g. Gradshteyn and Ryzhik 1980, p 21)

$$\sum_{n=0}^{\infty} (-1)^n x^{2n+1} = \frac{x}{1+x^2}, \qquad |x| < 1. \tag{37}$$

Let us recall that the closed form (36) is obtained from the power series (35) in decaying exponentials $g(\xi) = \exp(-K\xi)$. Since the expansion (35) is only convergent under the condition $ag < 1$, the closed form (36) at first sight seems to be only meaningful in the region $\xi > \xi_0$ ($\xi_0 = (\ln a)/K$). However, the closed form (36) can alternatively be expanded in a convergent power series in $1/ag$, under the condition $ag > 1$ (i.e. in the region $\xi < \xi_0$). Furthermore, taking the left and right limit for $\xi \to \xi_0$ in (36) leads to the same value $\sqrt{2}$. So, we may conclude that (36) is valid for the entire region $-\infty < \xi < +\infty$. It is worthwhile mentioning at this point that by starting from rising exponentials $g^{-1}(\xi) = \exp(K\xi)$ one would also arrive at the same closed form (36).

Finally, we rescale $\tilde{\phi}$ by (26) and return to the original variables using (21), (24) and (25), to obtain

$$u(x, t) = (6v/\alpha)^{1/2} \operatorname{sech}[\sqrt{v}(x - vt) + \delta], \tag{38}$$

where δ ($= \ln 1/a$) defines an arbitrary constant phase shift.

We have thus constructed the well known solitary wave solution of the mKdV equation (20) (Bullough and Caudrey 1980, Dodd *et al* 1982).

From a physical point of view, it is interesting to realise that a sech solution is only built up of odd harmonics of the fundamental function g, these obviously being the only ones to be generated by a cubic non-linearity.

3.2. *The KdV equation with additional fifth-order dispersion term*

An interesting example to illustrate the use of our technique in the case of multiple values of K and the consequent physical requirements of commensurability (see Hereman *et al* 1985) was originally found in Kodama and Tanuiti (1978) and Yamamoto and Takizawa (1981). The latter authors showed that the non-linear fifth-order differential equation

$$u_t + \alpha u u_x + \beta u_{3x} + u_{5x} = 0 \tag{39}$$

allows a sech^4-type solitary wave solution, for well defined values of the constant real numbers α and β.

It is our aim to solve equation (39) by our direct algebraic method, paying more attention to the mathematical details than to the physical interpretation, which has already been outlined in Hereman *et al* (1985).

Performing steps (a) and (b) of the method described in § 2 leads to

$$-v\phi+\tfrac{1}{2}\alpha\phi^2+\beta\phi_{2\xi}+\phi_{4\xi}=0, \tag{40}$$

where we neglected the integration constant to look only for solutions without a constant term c_1 in their expansion into real exponentials.

Clearly, the linear part of (40) (i.e. for $\alpha=0$) allows real exponential solutions $\exp[\pm K(v)\xi]$ for two different values K_1 and K_2, namely

$$K_{1,2}=\{\tfrac{1}{2}[-\beta\pm(\beta^2+4v)^{1/2}]\}^{1/2}, \tag{41}$$

with

$$\beta<0, \qquad -\tfrac{1}{4}\beta^2<v<0, \tag{42}$$

as sufficient conditions. Hence, $N=2$ and we define

$$g_{1,2}(\xi)\triangleq\exp(-K_{1,2}\xi). \tag{43}$$

Anticipating, now, that the final solution may be built up as a sum of powers of only one decaying exponential

$$g(\xi)\triangleq\exp(-K\xi), \tag{44}$$

we look for two integers $M_{1,2}$ satisfying

$$K=K_1/M_1=K_2/M_2. \tag{45}$$

For computational convenience that will become clear later, we rescale the coefficients in (40) by

$$\phi=-(v/18\alpha)\tilde{\phi} \tag{46}$$

and, thereafter, substitute the expansion

$$\tilde{\phi}=\sum_{n=1}^{\infty}a_n g^n \tag{47}$$

into the rescaled non-linear equation. This yields

$$\sum_{n=1}^{\infty}(n^4K^4+n^2\beta K^2-v)a_n g^n-\frac{v}{36}\sum_{n=2}^{\infty}\sum_{l=1}^{n-1}a_l a_{n-l}g^n=0, \tag{48}$$

where use has been made of Cauchy's rule for the double product appearing in $\tilde{\phi}^2$.

Since $K\neq K_1\neq K_2$, it follows from (48) with (41) that $a_1=0$. (The degenerate case $K_1=K_2$, i.e. $4v=-\beta^2$, is still under investigation.) For a non-trivial solution built up of the mixing of the two decaying exponentials $g_{1,2}$ as in (43), we require two coefficients a_n to be arbitrary. An obvious choice is a_2 and a_3, so that, from (48), the conditions

$$16K^4+4\beta K^2-v=0, \tag{49}$$

$$81K^4+9\beta K^2-v=0, \tag{50}$$

must be fulfilled. Solving for v and K in terms of β, we obtain

$$v=-36\beta^2/169, \tag{51}$$

$$K=(-\tfrac{1}{13}\beta)^{1/2}. \tag{52}$$

Hence, using (51) and (52) in (41), it follows from (45) that

$$M_1 = 2, \qquad M_2 = 3, \tag{53}$$

as expected on the basis of our choice ($g_1 = g^2$, $g_2 = g^3$). Hence

$$K_1 = 2(-\tfrac{1}{13}\beta)^{1/2}, \qquad K_2 = 3(-\tfrac{1}{13}\beta)^{1/2}. \tag{54}$$

Examining (51) and (52), we note that the conditions on β and v as stated in (42) are indeed fulfilled.

Simplifying (48) by using (51) and (52), we obtain the recursion relation

$$(n^2-4)(n^2-9)a_n + \sum_{l=1}^{n-1} a_l a_{n-l} = 0, \qquad n \geq 2, \tag{55}$$

justifying the scale in (46). Here again one can calculate the first few coefficients a_n from (55), leading to

$$a_4 = -a_2^2/84, \tag{56}$$

$$a_5 = -a_2 a_3/168, \tag{57}$$

$$a_6 = a_2^3/36288 - a_3^2/864, \tag{58}$$

etc, but it is very hard to speculate on what the explicit form of a_n will be. Therefore, we calculate the degree δ of the polynomial in a_n using (18), and we use the method described in point (2) of step (h) in § 2, together with the symmetry considerations cited there. For our example $\lambda = 4$ and $H = 2$, so $\delta = 3$. We remark that regarding (56)-(58) one may expect an alternation in sign in successive a_n if a_2 and a_3 have opposite signs. Furthermore, since $a_1 = 0$, $n-1$ is a factor in a_n, likewise $n+1$, the remaining factor being nothing else than n itself.

Finally, note that if a_n is a solution of (55) then $a_n a^n$, with $a > 0$ and constant, is also a solution of the same recursion relation. Taking all this into account, the form of a_n must be

$$a_n = b(-1)^{n+1} n(n-1)(n+1)a^n. \tag{59}$$

Now, the constants a and b, which may both depend on a_2 and a_3, must be determined. In order to calculate b we substitute (59) into (55) and we apply the formulae (Spiegel 1968)

$$S_1 = \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}, \tag{60}$$

$$S_2 = \sum_{i=1}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6}, \tag{61}$$

$$S_3 = \sum_{i=1}^{n-1} i^3 = \frac{(n-1)^2 n^2}{4}, \tag{62}$$

$$S_4 = \sum_{i=1}^{n-1} i^4 = \frac{(n-1)n(2n-1)(3n^2-3n-1)}{30}, \tag{63}$$

$$S_5 = \sum_{i=1}^{n-1} i^5 = \frac{(n-1)^2 n^2 (2n^2-2n-1)}{12}, \tag{64}$$

$$S_6 = \sum_{i=1}^{n-1} i^6 = \frac{(n-1)n(2n-1)(3n^4-6n^3+3n+1)}{42}, \tag{65}$$

for the sums of powers of the first $(n-1)$ positive integers. After straightforward algebra we obtain $b = 140$. Then $a = -a_3/4a_2 > 0$ follows readily from (59) expressed for $n = 2$ and $n = 3$, provided

$$a_2^3 = -\tfrac{105}{2} a_3^2. \tag{66}$$

After substitution of

$$a_n = 140(-1)^{n+1} n(n^2-1) a^n \tag{67}$$

into (47) and application of the formula (Gradshteyn and Ryzhik 1980) for the binomial series

$$\frac{x^2}{(1+x)^4} = \tfrac{1}{6} \sum_{n=2}^{\infty} (-1)^n n(n^2-1) x^n, \qquad |x| < 1, \tag{68}$$

$\tilde{\phi}$ can be re-expressed in closed form:

$$\tilde{\phi} = -840(ag)^2/(1+ag)^4. \tag{69}$$

For the same reasons as in §3.1 one can overcome restraints on the convergence region for (69), making this closed form valid for $-\infty < \xi < +\infty$.

Finally, returning to the original variables, using (44), (46), (51) and (52), we obtain

$$u(x,t) = \phi(x-vt) = -\frac{105\beta^2}{169\alpha} \operatorname{sech}^4\left[\frac{1}{2}\left(\frac{-\beta}{13}\right)^{1/2}\left(x + \frac{36\beta^2}{169} t\right) + \delta\right], \tag{70}$$

with δ $(=\tfrac{1}{2}\ln a^{-1})$ being an arbitrary phase shift. Equation (70) is the exact solution of (39) as obtained before by Yamamoto and Takizawa (1981).

While it was practical to calculate the solution (70) using a series expansion in a single exponential function g (the same closed form could have been obtained from an expansion in g^{-1}), in retrospect, one can expand (70) as a double series in g_1 and g_2. Indeed, note that (47) with (67) and $a = -a_3/4a_2$ may be re-expressed as

$$\tilde{\phi}(\xi) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} a_{n_1 n_2} g_1^{n_1}(\xi) g_2^{n_2}(\xi), \qquad (n_1, n_2) \neq (0,0), \tag{71}$$

with $g_1 = g^2$, $g_2 = g^3$ and $a_{10} = a_2$, $a_{01} = a_3$, $a_{20} = a_4 = -a_2^2/84$, $a_{02} = a_2^3/36\,288$, $a_{11} = a_5 = -a_2 a_3/168$, $a_{30} = -a_3^2/864$, etc, clearly demonstrating that the final solution is representable in terms of the harmonics and cross-couplings of the two (decaying) exponentials g_1 and g_2.

3.3. The non-linear Klein-Gordon equation

As an example of a non-linear wave equation let us consider the u^4 equation of particle physics (Dodd *et al* 1984, Narita 1984), commonly called the non-linear Klein-Gordon equation

$$u_{2t} - u_{2x} + \alpha u + \beta u^3 = 0, \tag{72}$$

where the constants α and β have opposite signs. We have chosen this equation since it has a constant term in the expansion of its kink-type solution into a power series in real exponentials.

As before we introduce the new variable $\xi = x - vt$ in (72), to obtain

$$(v^2-1)\phi_{2\xi} + \alpha\phi + \beta\phi^3 = 0. \tag{73}$$

In the most general case the solution of (73) can be written as

$$\phi = c_1 + \hat{\phi}, \tag{74}$$

where c_1 is a constant. Hence, $\hat{\phi}$ has to satisfy

$$(v^2-1)\hat{\phi}_{2\xi} + (\alpha + 3\beta c_1^2)\hat{\phi} + 3\beta c_1 \hat{\phi}^2 + \beta\hat{\phi}^3 + c_1(\alpha + \beta c_1^2) = 0. \tag{75}$$

Like non-dissipative evolution equations, the linear part of (75) has exponential solutions $\exp[\pm K(v)\xi]$, if

$$c_1(\alpha + \beta c_1^2) = 0. \tag{76}$$

For $c_1 = 0$ we find

$$K = [\alpha/(1-v^2)]^{1/2}, \tag{77}$$

from which it is clear that for $\alpha > 0$, $v^2 < 1$ and for $\alpha > 0$, $v^2 > 1$.
For $c_1^2 = -\alpha/\beta$, requiring that α and β have opposite signs,

$$K = [2\alpha/(v^2-1)]^{1/2}, \tag{78}$$

so that $\alpha > 0$ implies $v^2 > 1$.

For the first case, we perform the scaling

$$\phi = \hat{\phi} = \pm(-\alpha/\beta)^{1/2}\tilde{\phi}, \tag{79}$$

which is only possible if α and β have opposite signs. Next, we use (44), (47) and (77) to get the recursion relation

$$(n^2-1)a_n + \sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_l a_{m-l} a_{n-m} = 0, \qquad n \geq 3,\ a_1 \text{ arbitrary},\ a_2 = 0, \tag{80}$$

which is identical to the recursion relation (29) for the mKdV equation. Hence,

$$\tilde{\phi}(\xi) = 2\sqrt{2}\frac{ag(\xi)}{1 + a^2 g^2(\xi)}, \qquad a = \frac{a_1}{2\sqrt{2}} > 0, \tag{81}$$

so that we re-obtain the well known solitary wave solution (Narita 1984)

$$u(x, t) = \pm(-2\alpha/\beta)^{1/2}\,\mathrm{sech}\{[\alpha/(1-v^2)]^{1/2}(x - vt) + \delta\}, \tag{82}$$

with $\delta = \ln a^{-1}$.
For the second case, where

$$c_1 = \pm(-\alpha/\beta)^{1/2}, \tag{83}$$

we again use the scale transformation (79), and we substitute the series expansion for $\tilde{\phi}$ into the resulting equation. This yields

$$(n^2-1)a_n \mp \tfrac{3}{2}\sum_{l=1}^{n-1} a_l a_{n-l} - \tfrac{1}{2}\sum_{m=2}^{n-1}\sum_{l=1}^{m-1} a_l a_{m-l} a_{n-m} = 0, \qquad n \geq 3, \tag{84}$$

with a_1 arbitrary and $a_2 = a_1^2/2$. The solution of (84) is easily found to be proportional to a constant $\tilde{c}$, explicitly

$$a_n = \pm 2(\tfrac{1}{2}a_1)^n \tag{85}$$

so that

$$\tilde{\phi} = 2ag/(1 \mp ag), \qquad a = \tfrac{1}{2}a_1 > 0. \tag{86}$$

Again it can be shown that the closed form (86) is valid over the entire region $-\infty < \xi < \infty$.

For a physical solution we take the plus sign in (86); this means we have chosen opposite signs for c_1 in (83) and the scaling in (79). Returning to the original variables, we can express the final result in the form

$$u(x, t) = \pm(-\alpha/\beta)^{1/2} \tanh\{[\alpha/2(v^2-1)]^{1/2}(x - vt) + \delta\}, \tag{87}$$

with $\delta = \frac{1}{2}\ln a^{-1}$. Equation (87) is nothing else than the tanh-type kink (Dodd *et al* 1982, Narita 1984).

We may conclude that only slightly modifying our technique enables us to construct two different particular exact solutions of the non-linear Klein–Gordon equation (72).

3.4. The sine-Gordon equation

As a final example we focus on the single sine-Gordon equation

$$\sigma_{2\tau} - \sigma_{2\eta} = \sin \sigma, \tag{88}$$

arising in a variety of physical and mathematical contexts (Newell 1983). The (non-) linear approximation to (88) is the (non-)linear Klein–Gordon equation.

The coordinate transformation $\tau = x + t$, $\eta = x - t$ modifies (88) to a more suitable form for our purpose, i.e.

$$u_{xt} = \sin u, \tag{89}$$

for $u(x, t) \triangleq \sigma(\tau, \eta)$.

In order to avoid dealing with the transcendental non-linearity sin u, we introduce, after Rosales (1978), two new variables

$$\Phi = u_x, \tag{90}$$

$$\Psi = \cos u - 1. \tag{91}$$

Hence, (89) can be replaced by the following set of non-linear coupled equations:

$$\Phi_{xt} - \Phi - \Phi\Psi = 0, \tag{92}$$

$$2\Psi + \Psi^2 + \Phi_t^2 = 0. \tag{93}$$

Now, we show how our solution technique applies to this coupled system. As before we seek stationary travelling wave solutions of (92)-(93) in the form $\phi(\xi) \triangleq \Phi(x, t)$, $\psi(\xi) = \Psi(x, t)$, with $\xi = x - vt$. Upon substitution of the expansions of the normalised functions

$$\tilde{\phi} = \sqrt{-v}\,\phi = \sum_{n=1}^{\infty} a_n g^n(\xi), \tag{94}$$

$$\tilde{\psi} = \psi = \sum_{n=1}^{\infty} b_n g^n(\xi), \tag{95}$$

with $v < 0$, we obtain the following infinite set of coupled recursion relations:

$$(n^2 - 1)a_n - \sum_{l=1}^{n-1} a_l b_{n-l} = 0, \qquad n \geq 2,\ a_1 \text{ arbitrary}, \tag{96}$$

$$2b_n + \sum_{l=1}^{n-1} b_l b_{n-l} + \sum_{l=1}^{n-1} l(n-l)a_l a_{n-l} = 0, \qquad n \geq 2,\ b_1 = 0, \tag{97}$$

where we have used the relation $v=-1/k^2$ (obtained from the linear part of (92)) to simplify. By an iterative calculation of the first few coefficients, resulting in $a_2=a_4=a_6=\ldots=0$, $a_3=-a_1^3/2^4$, $a_5=a_1^5/2^8$, etc, it is easy to write down a non-trivial expression for a_n in the form

$$a_{2n'+1}=(-1)^{n'}a_1^{2n'+1}/2^{4n'}, \qquad n'\geq 0, \tag{98}$$

$$a_{2n'}=0, \qquad n'\geq 1. \tag{99}$$

One can calculate the b_n in a similar way.

Returning to (94) and proceeding as in the first example, we easily find ϕ in its closed form,

$$\phi=4kag/[1+(ag)^2], \qquad a=a_1/4>0. \tag{100}$$

After substituting (100) into (90) and integrating, we finally obtain the kink-profile solution (Lamb 1980, Pöppe 1983)

$$u(x,t)=\pm 4\tan^{-1}\{\exp[(x-vt)/\sqrt{-v}+\delta]\}, \tag{101}$$

for $k>0$ and $v<0$. Other kink and antikink solutions (not listed in table 1), arising from different combinations of the signs of k and v, can be derived in a similar way (cf Lamb 1980).

Other coupled systems of evolution equations (Kupershmidt 1984) or coupled wave equations (e.g. the Maxwell-Bloch equations (Dodd *et al* 1982)) may be found solvable by our method. The results for a coupled system of KdV equations (Hirota and Satsuma 1981, Satsuma and Hirota 1982) are listed in table 1. Detailed calculations will be presented elsewhere (Banerjee and Hereman 1985).

4. Stationary solutions of other non-linear evolution/wave equations

We have used our method to find single solitary wave solutions of a variety of non-linear evolution and wave equations. In table 1, we give a survey of these equations, together with the corresponding recursion relations and the final physically relevant solutions. For most cases, except the KdV, the Klein-Gordon and the second-order Benjamin-Ono equation, we did not incorporate c_1 in the solution, only to avoid unnecessary mathematical computations. The physical details on the construction of a (general) solitary wave solution of the KdV equation, the Burgers equation and the KdV equation with an additional fifth-order dispersion term are given by Hereman *et al* (1985).

For dissipative equations, e.g. the Burgers equation (Whitham 1974), where only one type of solution for the linear equation exists, our technique is still applicable in a straightforward way, though the physical interpretation of the closed form solution, in terms of mixing and harmonics, is considerably different.

The regularised long wave (RLW) and the time-regularised long wave (TRLW) equations (Benjamin *et al* 1972, Jeffrey 1978) can be dealt with in a similar way to the KdV equation itself. The Sharma-Tasso-Olver (STO) equation which appeared in Sharma and Tasso (1977), Olver (1977) and Verheest and Hereman (1982), had (to our knowledge) not been exactly solved before.

For comparison of our technique with standard methods, particular examples of fifth-order generalised KdV equations may be found in Lax (1968), Sawada and Kotera (1974), Ito (1980), Ablowitz *et al* (1974), Fordy and Gibbons (1980) and many others.

Table 1. Evolution and wave equations solved by the direct algebraic method.

Evolution/Wave Equation	Chosen c_1	#K	K	Recursion Relation	Solution	Solitary Wave Solution
Burgers $u_t + \alpha u u_x - u_{2x} = 0$	0	1	v	$(n-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto \tilde{c}$	$\frac{v}{\alpha}[1 - \tanh\frac{v}{2}(x-vt)+\delta]$
Korteweg-de Vries (KdV) $u_t + \alpha u u_x + u_{3x} = 0$	$\frac{3v}{2\alpha}$ 0	2 2	$\pm\sqrt{-\frac{v}{2}}$ $\pm\sqrt{v}$	$(n^2-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$ $(n^2-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto n$ $a_n \propto n$	$\frac{3v}{2\alpha}\tanh^2\frac{\sqrt{-\frac{v}{2}}}{2}(x-vt)+\delta$ $\frac{3v}{\alpha}\mathrm{sech}^2\frac{\sqrt{v}}{2}(x-vt)+\delta$
Benjamin-Bona-Mahony (RLW) $u_t + u_x + \alpha u u_x - u_{2xt} = 0$	0	2	$\pm\sqrt{\frac{v-1}{v}}$	$(n^2-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$\frac{3(v-1)}{\alpha}\mathrm{sech}^2\frac{1}{2}\sqrt{\frac{v-1}{v}}(x-vt)+\delta$
Joseph-Egri (TRLW) $u_t + u_x + \alpha u u_x + u_{x2t} = 0$	0	2	$\pm\frac{\sqrt{v-1}}{v}$	$(n^2-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$\frac{3(v-1)}{\alpha}\mathrm{sech}^2\frac{\sqrt{v-1}}{2v}(x-vt)+\delta$
Modified Korteweg-de Vries (mKdV) $u_t + \alpha u^2 u_x + u_{3x} = 0$	0	2	$\pm\sqrt{v}$	$(n^2-1)a_n + \sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m} = 0$	$a_{2n} = 0$ $a_{2n+1} \propto \tilde{c}$	$\sqrt{\frac{6v}{\alpha}}\,\mathrm{sech}\sqrt{v}\,(x-vt)+\delta$
Sharma-Tasso-Olver (STO) $u_t + 3u_x^2 + 3u^2 u_x + 3u u_{2x} + u_{3x} = 0$	0	2	$+\sqrt{v}$	$(n^2-1)a_n \mp 3\sum_{\ell=1}^{n-1}\ell a_\ell a_{n-\ell} + \sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m} = 0$	$a_n \propto \tilde{c}$	$\frac{\sqrt{v}}{2}[\tanh^{\mp1}(\frac{\sqrt{v}}{2}(x-vt)+\delta)\mp 1]$
KdV with additional fifth order dispersion term $u_t + \alpha u u_x + \beta u_{3x} + u_{5x} = 0$	0	4	$\pm 2\sqrt{\frac{-\beta}{13}}$ $\pm 3\sqrt{\frac{-\beta}{13}}$	$(n^2-4)(n^2-9)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto n^3$	$\frac{-105\beta^2}{169\alpha}\mathrm{sech}^4\frac{1}{2}\sqrt{\frac{-\beta}{13}}(x+\frac{36\beta^2}{169}t)+\delta$
Fifth order generalized KdV $u_t + \alpha u^2 u_x + (\frac{10\alpha-\gamma^2}{\gamma})u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0$	0	2	$\pm\sqrt[4]{v}$	$(n^4-1)a_n + \frac{\alpha}{3}\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m}$ $+ \frac{1}{\gamma}\sum_{\ell=1}^{n-1}\ell[(5\alpha-\gamma^2)n-(5\alpha-2\gamma^2)\ell]a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$\frac{3\gamma}{2\alpha}\sqrt{v}\,\mathrm{sech}^2\frac{\sqrt[4]{v}}{2}(x-vt)+\delta$
Fifth order dispersive equation $u_t + \beta u_x u_{2x} + u_{5x} = 0$	0	2	$\pm\sqrt[4]{v}$	$(n^4-1)a_n + \sum_{\ell=1}^{n-1}\ell(n-\ell)a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$\frac{15}{\beta}\sqrt{v}\,\mathrm{sech}^2\frac{1}{2}\sqrt[4]{v}(x-vt)+\delta$
Seventh order generalized KdV $u_t + \alpha u^3 u_x + 2(\gamma+\tilde{\delta})u u_x u_{2x}$ $+ \gamma u^2 u_{3x} + \tilde{\delta} u_x^3 + \epsilon u_x u_{4x}$ $+ \zeta u u_{5x} + \eta u_{2x} u_{3x} + \theta u_{7x} = 0$	0	2	$\pm\sqrt[6]{\frac{v}{\theta}}$	$(n^6-1)a_n + \frac{\alpha}{4}\sum_{m=3}^{n-1}\sum_{\ell=2}^{m-1}\sum_{p=1}^{\ell-1} a_p a_{\ell-p} a_{m-\ell} a_{n-m}$ $+ K^2\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1}\ell[\gamma\ell+\tilde{\delta}(n-m)]a_\ell a_{m-\ell} a_{n-m}$ $+ \frac{K^4}{2}\sum_{\ell=1}^{n-1}\ell^2[(5\zeta-2\epsilon+\eta)\ell^2 + 2(2\epsilon-2\zeta-\eta)n\ell$ $+ (\eta-\epsilon+\zeta)n^2]a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$A\sqrt[3]{\frac{v}{\theta}}\,\mathrm{sech}^2\frac{1}{2}\sqrt[6]{\frac{v}{\theta}}(x-\frac{ABv}{\theta}t)+\delta$ $A = (5\epsilon+20\zeta-\eta)/6(\gamma+\tilde{\delta})$ $B = (\epsilon+\zeta+\eta)/63$
Kadomtsev-Petviashvili (KP) & Two-dimensional KdV ($\epsilon < 0$ resp. $\epsilon > 0$) $u_{tx} + \alpha u_x^2 + \alpha u u_{2x} + \gamma u_{4x} + \epsilon u_{2y} = 0$ $\xi = x - vt + wy$	0	2	$\pm\sqrt{\frac{v-w^2\epsilon}{\gamma}}$	$(n^2-1)a_n + \sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} = 0$	$a_n \propto n$	$\frac{3(w^2\epsilon-v)}{\alpha}\mathrm{sech}^2\frac{1}{2}\sqrt{\frac{v-w^2\epsilon}{\alpha}}(x-vt)+\delta$
Nonlinear Klein-Gordon (NKG) $u_{2t} - u_{2x} + \alpha u + \beta u^3 = 0$ $\alpha\beta < 0$	0 $+\sqrt{-\frac{\alpha}{\beta}}$	2 2	$+\sqrt{\frac{\alpha}{1-v^2}}$ $\pm\sqrt{\frac{2\alpha}{v^2-1}}$	$(n^2-1)a_n + \sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m} = 0$ $(n^2-1)a_n - \frac{3}{2}\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell} - \frac{1}{2}\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m} = 0$	$a_{2n} = 0$ $a_{2n+1} \propto \tilde{c}$ $a_{2n} \propto \tilde{c}$	$+\sqrt{\frac{-2\alpha}{\beta}}\,\mathrm{sech}\sqrt{\frac{\alpha}{1-v^2}}(x-vt)+\delta$ $\pm\sqrt{\frac{-\alpha}{\beta}}\tanh\sqrt{\frac{\alpha}{2(v^2-1)}}(x-vt)+\delta$

Boussinesq (BE) & Good BE ($\beta > 0$, resp. $\beta < 0$) $u_{2t}-u_{2x}+\alpha u_x^2+\alpha u u_{2x}-\beta u_{4x}=0$	0	2	$\pm\sqrt{\frac{v^2-1}{\beta}}$	$(n^2-1)a_n+\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell}=0$	$a_n \propto n$	$\frac{3(1-v^2)}{\alpha}\,\mathrm{sech}^2\,\frac{1}{2}\sqrt{\frac{v^2-1}{\beta}}(x-vt)+\delta$
Improved BE ($\beta > 0$) $u_{2t}-u_{2x}+\alpha u_x^2+\alpha u u_{2x}+\beta u_{2x2t}=0$	0	2	$\pm\sqrt{\frac{1-v^2}{\beta v^2}}$	$(n^2-1)a_n+\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell}=0$	$a_n \propto n$	$\frac{3(1-v^2)}{\alpha}\,\mathrm{sech}^2\,\frac{1}{2}\sqrt{\frac{1-v^2}{\beta v^2}}(x-vt)+\delta$
Modified Improved BE $u_{2t}-u_{2x}+2\alpha u u_x^2+\alpha u^2 u_{2x}$ $+\beta u_{2x2t}=0$ $\alpha < 0\ ,\ \beta > 0$	0	2	$\pm\sqrt{\frac{1-v^2}{\beta v^2}}$	$(n^2-1)a_n+\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-\ell}=0$	$a_{2n}=0$ $a_{2n+1}\propto \tilde{c}$	$\sqrt{\frac{6(1-v^2)}{\alpha}}\,\mathrm{sech}\sqrt{\frac{1-v^2}{\beta v^2}}(x-vt)+\delta$
Thomas* $u_{xy}+\alpha u_x+\beta u_y+u_x u_y=0$	0	1	$\frac{\alpha-\beta v}{v}$	$n(n-1)a_n+\sum_{\ell=1}^{n-1}\ell(n-\ell)a_\ell a_{n-\ell}=0$	$a_n \propto \frac{1}{n}$	$\ln[1+\exp\frac{\alpha-\beta v}{v}(x-vy)+\delta]$
Second order Benjamin Ono $u_{2t}+\alpha(u^2)_{2x}+\beta u_{4x}=0$	$-\frac{3v^2}{4\alpha}$ 0	1 1	$\frac{\pm v}{\sqrt{2\beta}}$ $\frac{\pm v}{\sqrt{-\beta}}$	$(n^2-1)a_n+\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell}=0$ $(n^2-1)a_n+\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell}=0$	$a_n \propto n$ $a_n \propto n$	$-\frac{3v^2}{4\alpha}\,\tanh^2\,\frac{v}{2\sqrt{2\beta}}(x-vt)+\delta$ $-\frac{3v^2}{2\alpha}\,\mathrm{sech}^2\,\frac{v}{2\sqrt{-\beta}}(x-vt)+\delta$

For Thomas' equation (neither an evolution nor a wave equation) we take $\xi = k(x-vy)$.

mKdV with additional first order dispersion term $u_t-\alpha u^2 u_x+u_{3x}+\beta u_x=0$	0	1	$\pm\sqrt{v-\beta}$	$(n^2-1)a_n-\sum_{m=2}^{n-1}\sum_{\ell=1}^{m-1} a_\ell a_{m-\ell} a_{n-m}=0$	$a_n \propto \tilde{c}$	$\pm\sqrt{\frac{6(v-\beta)}{\alpha}}\,\mathrm{cosech}\sqrt{v-\beta}(x-vt)+\delta$
Sine-Gordon $u_{2t}-u_{2x}=\sin u$	0	1	$\frac{1}{\sqrt{v}}$	$(n^2-1)a_n-\sum_{\ell=1}^{n-1} a_\ell b_{n-\ell}=0$ $2b_n+\sum_{\ell=1}^{n-1} b_\ell b_{n-\ell}+\sum_{\ell=1}^{n-1}\ell(n-\ell)a_\ell a_{n-\ell}=0$	$a_{2n}=0$ $a_{2n+1}\propto\tilde{c}$ $b_{2n+1}=0$ $b_{2n}\propto n$	$\pm\arctan[\exp\frac{(1-v)}{2\sqrt{v}}(x-\frac{v+1}{v-1}t)+\delta]$
Hirota's shallow water wave equation $\lambda u_t-u_{xxt}+3u_x(1-u_t)=0$	0	1	$\pm\sqrt{\frac{\lambda v-3}{v}}$	$n(n^2-1)a_n+\sum_{\ell=1}^{n-1}\ell(n-\ell)a_\ell a_{n-\ell}=0$	$a_n \propto \tilde{c}$	$\sqrt{\frac{\lambda v-3}{v}}[1\pm\tanh\frac{1}{2}\sqrt{\frac{\lambda v-3}{v}}(x-vt)+\delta]$
Coupled KdV $u_t-\alpha(u_{3x}+6uu_x)-2\beta w w_x=0$ $w_t+w_{3x}+3uw_x=0$	0	1	$\sqrt{v}$	$(\alpha n^2+1)a_n+3\alpha\sum_{\ell=1}^{n-1} a_\ell a_{n-\ell}+\beta\sum_{\ell=1}^{n-1} b_\ell b_{n-\ell}=0$ $n(n^2-1)b_n+3\sum_{\ell=1}^{n-1}\ell b_\ell a_{n-\ell}=0$	$a_{2n}\propto n$ $a_{2n+1}=0$ $b_{2n}=0$ $b_{2n+1}\propto\tilde{c}$	$u = 2v\,\mathrm{sech}^2\sqrt{v}(x-vt)+\delta$ $w = Av\,\mathrm{sech}\sqrt{v}(x-vt)+\delta$ $A^2=\frac{-2(1+4\alpha)}{\beta}$

The first three papers listed here also discuss two particular examples of the seventh-order generalised KdV equations mentioned in table 1. The special and drastically simplified version of a fifth-order dispersive equation was solved by Van Immerzeele (1983) using a preliminary version of our method. The Kadomtsev–Petviashvili (KP) and the two-dimensional KdV equations can be found in e.g. Kadomtsev and Petviashvili (1970), Satsuma (1976) and Freeman (1980). The Boussinesq equation (BE) is discussed by many authors, including Whitham (1974) and Scott *et al* (1973). Details on the 'good' BE, the improved BE and the modified improved BE, respectively, may be obtained from McKean (1981), Iskandar and Jain (1980) and Bogolubsky (1977). Note that the recursion relations for all the mentioned BEs (except the modified improved one) and the KP equation are exactly the same as the recursion relation corresponding to the KdV equation, and that the ones corresponding to the mKdV and the modified improved BE are identical.

In addition to evolution and wave equations, our technique can also be applied to other non-linear PDEs. For instance, for the Thomas equation (Rosales 1978, Roy Chowdhury and Paul 1984), our method yields a ln-type solution. For a second-order Benjamin–Ono equation (Korpel and Banerjee 1984) it leads to a well-type solution; for a mKdV with an additional first-order dispersion term (Fung and Au 1984) we obtain a cosech solution. Detailed calculations for the last three equations can be found in Meerpoel (1985). A model equation for shallow water waves (Hirota and Satsuma 1976, Hirota and Ramani 1980) has also been successfully solved by our method. In table 1 we list its potential form (obtained after integration and scaling) with a corresponding solution.

Furthermore, coupled systems can also be treated by our method. As we have seen in § 3, the sine-Gordon equation and perhaps other equations with transcendental non-linearities (e.g. Liouville's equation (Bullough and Caudrey 1980), Mikhailov's equations (Hirota and Ramani 1980), double sine-Gordon equations (Bullough and Caudrey 1980)), which appear not to be solvable as they stand, may be transformed into a solvable coupled set of non-linear equations. As a final example, we list a coupled system of KdV equations (Hirota and Satsuma 1981) and a particular solution.

5. Construction of N-solitary wave solutions—comparison with other methods

In this section, we outline how our method can be extended in order to construct multiple solitary wave solutions, taking the KdV equation as an example. Since we are looking for solutions of N solitons, each moving with a different velocity and having different amplitudes and widths in general, we can no longer choose a single travelling frame of reference and integrate. We therefore start from

$$u_t + \alpha u u_x + u_{3x} = 0, \qquad (102)$$

with α an arbitrary real number.

The linear part (found by setting $\alpha = 0$) admits a solution of the form

$$u^{(1)} = \sum_{i=1}^{N} c_i g_i(x, t), \qquad g_i \triangleq a_i \exp(k_i x - \omega_i t), \qquad (103)$$

where $c_i(k_i)$ and a_i are constants, and with the dispersion law

$$\omega_i = k_i^3, \qquad i = 1, \ldots, N. \qquad (104)$$

It will turn out later that the significance of there existing N linear solutions, rather than one as before, lies in the fact that ultimately the evolution by mixing results in N solitons. The superscript (1) in (103) refers to the finite single sum over i, i.e. the non-mixed terms. Higher-order mixing products will subsequently be denoted by $u^{(2)}$, $u^{(3)}$, etc. Note that in Rosales (1978) and Wadati and Sawada (1980a), the index would refer to a term in the formal perturbation expansion.

For the non-linear term αuu_x in (102) we write

$$\alpha u^{(1)}u_x^{(1)} = \tfrac{1}{2}\alpha[u_x^{(1)}(k_i)u^{(1)}(k_j) + u^{(1)}(k_i)u_x^{(1)}(k_j)], \tag{105}$$

leading to a double sum

$$\tfrac{1}{2}\alpha \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j (k_i + k_j) g_i g_j \tag{106}$$

similar to the non-linear term(s) in (16) for a quadratic non-linearity and for $n = 2$.

The analogue of $P(2)$ in the linear part of (16), in our case, results from the action of the linear operator

$$L. \triangleq \partial./\partial t + \partial^3./\partial_x^3 \tag{107}$$

on

$$u^{(2)} = \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} g_i g_j, \tag{108}$$

so that, using (104),

$$\begin{aligned} Lu^{(2)} &= \sum_{i=1}^{N}\sum_{j=1}^{N} L(\omega_i + \omega_j, k_i + k_j) c_{ij} g_i g_j \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} [-(\omega_i + \omega_j) + (k_i + k_j)^3] c_{ij} g_i g_j \\ &= 3\sum_{i=1}^{N}\sum_{j=1}^{N} k_i k_j (k_i + k_j) c_{ij} g_i g_j \end{aligned} \tag{109}$$

must replace the linear term $P(2)a_2 g^2$ in (16).

To arrive at the proper closed form for the N-soliton solution we equate the coefficients of terms that exhibit identical sequences of indices. (The physical reason for doing so in exactly this way is still under investigation, although we believe that it has to do with the symmetry properties of the asymptotic behaviour of the soliton sequence.) Proceeding in that manner, starting from

$$Lu^{(2)} + \tfrac{1}{2}\alpha(u_x^{(1)}u^{(1)} + u^{(1)}u_x^{(1)}) = 0, \tag{110}$$

and using (106) and (109), we readily obtain

$$c_{ij} = -\tfrac{1}{6}\alpha, \qquad i, j = 1, \ldots, N, \tag{111}$$

for the convenient choice

$$c_i = k_i, \qquad i = 1, \ldots, N. \tag{112}$$

It should be clear that from our physical point of view the double sum in (108) contains all possible mixing contributions between (k_i, ω_i) and (k_j, ω_j) for $i, j = 1, \ldots, N$.

In a similar way, the analogues of the linear and non-linear terms in (16) become, respectively,

$$Lu^{(n)} = \underbrace{\sum_{i=1}^{N} \sum_{j=1}^{N} \dots \sum_{s=1}^{N}}_{n \text{ summations}} L(\omega_i + \omega_j + \dots + \omega_s, k_i + k_j + \dots + k_s) c_{ij\dots s} g_i g_j \dots g_s \tag{113}$$

and

$$\frac{\alpha}{2} \sum_{l=1}^{n-1} u_x^{(l)}\underbrace{(k_i, k_j, \dots, k_p)}_{l \text{ arguments}} u^{(n-l)}\underbrace{(k_q, k_r, \dots, k_s)}_{n-l \text{ arguments}}$$
$$+ u^{(l)}(k_i, k_j, \dots, k_p) u_x^{(n-l)}(k_q, k_r, \dots, k_s). \tag{114}$$

The sum of (113) and (114) has to equal zero, from which $c_{ij\dots s}$ can be determined following the same index ordering procedure as used before.

After some algebra (Rosales 1978, Wadati and Sawada 1980a) one finds

$$c_{ij\dots s} = (-1)^{n-1} \left(\frac{\alpha}{6}\right)^{n-1} \frac{(k_i + k_j + \dots + k_r + k_s)}{(k_i + k_j)(k_j + k_k) \dots (k_r + k_s)}, \qquad i, j, \dots, s = 1, \dots, N. \tag{115}$$

Hence, like the nth harmonic used in the one-soliton case, the n-fold mixing contribution will be written as

$$u^{(n)} = \sum_{i=1}^{N} \sum_{j=1}^{N} \dots \sum_{s=1}^{N} c_{ij\dots s} g_i g_j \dots g_s. \tag{116}$$

Analogous to the way we build up a single soliton from simple harmonics in (6), we build up the N-soliton sequence from all the higher-order mixing terms. Hence, the solution is

$$u(x, t) = \sum_{n=1}^{\infty} u^{(n)}, \tag{117}$$

which may be re-expressed, using (115) and (116), as

$$u(x, t) = \sum_{i=1}^{N} k_i g_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \left(-\frac{\alpha}{6}\right) g_i g_j + \dots + \sum_{i=1}^{N} \sum_{j=1}^{N} \dots \sum_{s=1}^{N} (-1)^{n-1} \left(\frac{\alpha}{6}\right)^{n-1}$$
$$\times \frac{(k_i + k_j + \dots + k_s)}{(k_i + k_j)(k_j + k_k) \dots (k_r + k_s)} g_i g_j \dots g_s + \dots . \tag{118}$$

It can now be shown (Wadati and Sawada 1980a) that (118) can be written in closed form as

$$u(x, t) = (12/\alpha)(\partial^2/\partial x^2) \log \det(I + B), \tag{119}$$

where I refers to the $N \times N$ identity matrix and where the $N \times N$ matrix B has elements

$$B_{ij} = \frac{\alpha}{6} \frac{(a_i a_j)^{1/2}}{(k_i + k_j)} \exp \tfrac{1}{2}[(k_i + k_j)x - (\omega_i + \omega_j)t]. \tag{120}$$

The solution (119) is identical to the discrete N-soliton solution constructed using Hirota's bilinear formalism (Hirota 1971) or the inverse scattering technique (Gardner *et al* 1967). A Fourier integral representation of the solution may be found in Rosales' paper (Rosales 1978).

The N-soliton solutions of other evolutions and wave equations may be obtained in a similar way.

6. Discussion

One might wonder if solutions other than those expressible in a series expansion of real exponentials can be obtained with our method. In particular, we think of the algebraic solution of the Benjamin-Ono equation (Kodama *et al* 1982). Furthermore, it also seems interesting to try to treat difference-differential equations such as the equations of motion of an exponential lattice (i.e. Toda lattice (Toda 1981)) with our approach.

At the present stage, we have, concurrently with our physical interpretation, provided a formalism for systematically solving recurrence relations leading to single solitary wave solutions. Although our physical picture of N-soliton evolution is the same as in the one-soliton case, a generalisation of the formalism is still lacking here. We are also investigating how our method must be modified to treat complex, higher-dimensional or vector non-linear evolution and wave equations, or PDEs of a different kind.

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